Bonn International Graduate School in Mathematics

Multiple Zeta Values and the Connes-Moscovici Hopf Algebra

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part 1

Multiple Zeta Values

For an n-tuple of positive integers $\mathbf{k} = (k_1, k_2 \ldots, k_n)$ with $k_1 > 1$, we define multiple zeta values $\zeta(\mathbf{k}) = \zeta(k_1, k_2, \ldots, k_n)$ by the convergent series

$$\sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}},$$

MZV has an integral representaion as follows

$$\zeta(\mathbf{k}) = \int \cdots \int_{1 > t_1 > t_2 > \cdots > t_k > 0} \omega_1(t_1)\omega_2(t_2)\cdots\omega_k(t_k),$$

where $k = k_1 + k_2 + ... + k_n$ is the weight of $\zeta(\mathbf{k})$ and $\omega_i(t) = dt/(1-t)$ if $i \in \{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_n\}$ and $\omega_i(t) = dt/t$ otherwise.
There is a little known about arithmetical properties of these numbers. On the other hand there is a large amount of relations (linear and algebraic) among **MZV's**. One of the fascinating features of **MZV's** is that the structure of relations over $\mathbb{Q}$ reflects other structures in mathematics and physics.
It is due to Euler that $\zeta(2,1) = \zeta(3)$. By the very definition, it is clear that the product of two zeta values is a linear combination of multiple zeta values with integer coefficients so the the space spaned by all **MZV's** over the rationals form an algebra. In particular

$$\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b).$$

Let $\mathfrak{h} = \mathbb{Q}\langle x, y\rangle$ be the free algebra in two noncommutative variables x and y over the rationals and $\mathfrak{h}^0$ its subalgebra $\mathbb{Q} + x\mathfrak{h}y$. Define the linear map $\mathbf{Z}$ by the following assignment

$$\mathbf{Z}(x^{k_1-1}yx^{k_2-1}y \cdots x^{k_n-1}y) = \zeta(k_1, k_2, \ldots, k_n).$$

Let $z_k = x^{k-1}y$ and define the harmonic $*$ and the shuffle $\sqcup\!\sqcup$ products on $\mathfrak{h}$ by

$$1 * w = w * 1 = w, \quad z_k w_1 * z_l w_2 = z_k(w_1 * z_l w_2) + z_l(z_k w_1 * w_2) + z_k + l()w_1 * w_2,$$

$$1 \sqcup\!\sqcup w = w \sqcup\!\sqcup 1, \quad uw_1 \sqcup\!\sqcup vw_2 = u(w_1 \sqcup\!\sqcup vw_2) + v(uw_1 \sqcup\!\sqcup w_2).$$

The harmonic and the shuffle products correspond to the multiplication of **MZV's** represented as the convergent series and iterated integrals respectively. For any w_1 and w_2 in $\mathfrak{h}^0$ they are defined in such a way that we have

$$\mathbf{Z}(w_1)\mathbf{Z}(w_2) = \mathbf{Z}(w_1 * w_2), \quad \mathbf{Z}(w_1)\mathbf{Z}(w_2) = \mathbf{Z}(w_1 \sqcup\!\sqcup w_2).$$

This gives the double shuffle relations:

$$\mathbf{Z}(w_1 * w_2) = \mathbf{Z}(w_1 \sqcup\!\sqcup w_2).$$

part 2

The Modular Algebra

Let $\mathcal{M}$ be the ring of modular forms of all levels and all weights. $\mathcal{M}$ is graded by the weight and the group $\mathrm{PGL}_2^+(\mathbb{Q})$ acts on $\mathcal{M}$ via the slash operator:

$$f|_\gamma(z) = (det\gamma)^{k/2}(cz+d)^{-k} f(\gamma z),$$

where k is the weight of f, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2^+(\mathbb{Q})$ and $\gamma z = \frac{az+b}{cz+d}$. The crossed product $\mathcal{A} = \mathcal{M} \rtimes \mathrm{PGL}_2^+(\mathbb{Q})$ is called the modular algebra.

$\mathcal{M}$ is a differential algebra with respect to the Serre derivative, which is defined as

$$\vartheta : \mathcal{M}_k \longrightarrow \mathcal{M}_{k+2},$$

$$\vartheta(f) = \frac{1}{2\pi i} f' - \frac{k}{12} E_2 f,$$

where $E_2(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}$ is the Eisenstein series of weight 2 on $\mathrm{SL}_2(\mathbb{Z})$.

The Connes-Moscovici Hopf Algebra

Definition. $\mathcal{H}_1$ is the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n; n \geq 1\}$ and the brackets

$$[Y, X] = X, \ [Y, \delta_n] = n\delta_n, \ [X, \delta_n] = \delta_{n+1}.$$

The coproduct is given by multiplicativity and

$$\Delta Y = Y \otimes 1 + 1 \otimes Y, \ \Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \ \Delta\delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1,$$

the antipode is the anti-isomorphism given by

$$S(Y) = -Y, \ S(X) = -X + \delta_1 Y, \ S(\delta_1) = -\delta_1,$$

and the counit is

$$\varepsilon(h) = \text{ the constant term of } h \in \mathcal{H}_1.$$

part 3

CM-action

Theorem. $\mathcal{H}_1$ acts on $\mathcal{A}$ as follows:

$$Y(f \otimes \gamma) = \frac{\omega(f)}{2} f \otimes \gamma,$$

$$\delta_1(f \otimes \gamma) = \mu_\gamma f \otimes \gamma,$$

$$X(f \otimes \gamma) = \vartheta(f) \otimes \gamma,$$

$$\delta_{n+1}(f \otimes \gamma) = X\delta_n(f \otimes \gamma) - \delta_n X(f \otimes \gamma),$$

where $\omega(f)$ is the weight of f and the cocycle $\mu \in Z^1(\mathrm{PGL}_2^+(\mathbb{Q}), \mathcal{M}_2)$ is defined by

$$\mu_\gamma = \frac{1}{6}(E_2|_\gamma - E_2 - \frac{6}{\pi i}\frac{c}{cz+d}).$$

One can easily prove that $\delta_n(f \otimes \gamma) = \vartheta^{n-1}(\mu_\gamma) \otimes \gamma$.

The fact that above operators really determine a Hopf action of $\mathcal{H}_1$ on $\mathcal{A}$ is mostly a consequence of the following lemma.

Lemma. Let f be a modular form then

$$\vartheta(f|_\gamma) = \vartheta(f)|_\gamma + \frac{\omega(f)}{2}\mu_\gamma f|_\gamma.$$

Relations among MZV's

Some relations among **MZV's** can be shortly stated by the help of some derivations of $\mathfrak{h}$. For each integer number $n \geq 1$ define the derivation ∂_n on $\mathfrak{h}$ by

$$\partial_n(x) = x(x+y)^n y, \ \partial_n(y) = -x(x+y)^n y.$$

The derivation ∂_n has the following alternative description.
Let θ be the derivation on $\mathfrak{h}$ defined by

$$\theta(x) = x^2 + \frac{xy + yx}{2}, \ \theta(y) = y^2 + \frac{xy + yx}{2},$$

then one has $\partial_n = \frac{ad(\theta)^{n-1}\partial_1}{(n-1)!}$.

Theorem. One has $\mathbf{Z}(\partial_n(w)) = 0$ for all integers $n \geq 1$ and all $w \in \mathfrak{h}^0$.

part 4

Connection to MZV's

Let $\mathbf{L}_0$ be the quotient of the free Lie algebra on the letters X, ∂ by the relations that all $ad(X)^n(\partial)$ commute. Alternatively, $\mathbf{L}_0$ is the Lie algebra with basis $(X, \partial_1, \partial_2, \cdots)$ and relations $[X, \partial_n] = n\partial_{n+1}, [\partial_m \partial_n] = 0$. Since $\mathbf{L}_0$ is graded it naturally extends to the Lie algebra $\mathbf{L}_1 = \mathbf{L}_0 + \mathbb{Q}H$ where $[H, X] = X, [H, \partial_n] = n\partial_n$. There are several natural copies of $\mathbf{L}_1$ in the $\mathbf{Der}(\mathfrak{h})$. Clearly, $\mathbf{L}_1$ is the same Lie algebra as the underlying Lie algebra of the **CM** Hopf algebra. It would be very interesting to understand what lies behind this similarity which appears in two different subjects. Inspired by the **CM** Hopf algebra, M. Kaneko made the following conjecture.
Let H be the Euler operator on $\mathfrak{h}$ which send every homogeneous element w to $deg(w)w$. Let c be a rational number. For any positive integer n define a linear endomorphism $\partial_n^{(c)}$ by

$$\partial_n^{(c)} = \frac{ad(\theta^c)^{n-1}(\partial_1)}{(n-1)!},$$

where $\theta^{(c)}$ is defined by

$$\theta^{(c)}(x) = \theta(x), \quad \theta^{(c)}(y) = \theta(y), \quad \text{and } \theta^{(c)}(uv) = \theta^{(c)}(u)v + u\theta^{(c)}(v) + c\partial_1(u)H(v).$$

Conjecture. For all $n \geq 1$ and $c \in (Q)$ and any $w \in \mathfrak{h}^0$ one has

$$\mathbf{Z}(\partial_n^{(c)}(w)) = 0.$$

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