



Complex multiplication values of higher Green's functions

Mellit, Anton

advisor : Prof. Dr. D. Zagier

Bonn International Graduate School in Mathematics

part 1

1 Green's functions

Let \mathfrak{H} be the upper half plane. Let Γ be a congruence subgroup of $PSL_2(\mathbb{Z})$ and $k > 1$. The Green's function on \mathfrak{H}/Γ of weight $2k$ is the unique function G_k^Γ with the following properties:

- G_k^Γ is a smooth function on $\mathfrak{H} \times \mathfrak{H} - \{z_1 = \gamma z_2 \mid \gamma \in \Gamma\}$ with values in \mathbb{R} .
- $G_k^\Gamma(\gamma_1 z_1, \gamma_2 z_2) = G_k^\Gamma(z_1, z_2)$ for all $\gamma_1, \gamma_2 \in \Gamma$.
- $\Delta_i G_k^\Gamma = k(1-k)G_k^\Gamma$ (Δ_i is the Laplacian with respect to z_i).
- $G_k^\Gamma = \log|z_1 - z_2|^2 + O(1)$ when z_1 tends to z_2 .
- G_k^Γ tends to 0 when z_1 tends to a cusp.

2 Conjecture

A particular case of the conjecture formulated in [1] is as follows. Let $z_1, z_2 \in \mathfrak{H}$ be two different complex multiplication points with discriminants D_1, D_2 . Let $k = 2, 3, 4, 5, 7$. The conjecture says: There exists an algebraic number f such that the value of the Green's function equals

$$\tilde{G}_k^{PSL_2\mathbb{Z}}(z_1, z_2) = (D_1 D_2)^{\frac{1-k}{2}} \log f. \quad (*)$$

3 Examples

We have several examples which are verified numerically:

$$\begin{aligned} G_2^{PSL_2\mathbb{Z}}(\sqrt{-1}, \frac{-1 + \sqrt{-7}}{2}) &= -\frac{16}{\sqrt{28}} \log(8 + 3\sqrt{7}), \\ G_2^{PSL_2\mathbb{Z}}(\sqrt{-1}, \frac{-1 + \sqrt{-3}}{2}) &= -\frac{24}{\sqrt{12}} \log(2 + \sqrt{3}), \\ G_2^{PSL_2\mathbb{Z}}(\frac{-1 + \sqrt{-7}}{2}, \sqrt{-2}) &= \frac{4}{\sqrt{56}} \log\left(\frac{\pi_5 \pi_{13} u^{-5}}{\pi_5' \pi_{13}}\right), \text{ where} \\ \pi_5 &= 3 + \sqrt{14}, \pi_{13} = 1 + \sqrt{14}, u = 15 + 4\sqrt{14}. \end{aligned}$$

part 2

4 Main idea

The main idea is already present in the paper [2]. Suppose we have a family of elliptic curves over $\mathcal{E} \rightarrow S = \mathfrak{H}/\Gamma$. Then we consider

$$\mathcal{X} = \underbrace{\mathcal{E} \times_S \mathcal{E} \times_S \cdots \times_S \mathcal{E}}_{2k-2 \text{ times}}.$$

For a CM point z_1 we have the canonical cycle in the fibre of \mathcal{X} over z_1 , $\xi_{z_1} \in Z^{k-1}(X_{z_1})$. Suppose we have proved that ξ_{z_1} is trivial in the group $CH^k(\mathcal{X})$, that is we have found subvarieties $W_i \subset \mathcal{X}$ and rational functions $f_i \in \mathbb{Q}(W_i)$ such that

$$\xi_{z_1} = \sum_i \text{div } f_i.$$

For another CM point z_2 we intersect the canonical cycle ξ_{z_2} with W_i and form the product

$$f = \prod_i \prod_{p \in W_i} f_i(p)^{\text{ord}_{\xi_{z_2}} W_i p}.$$

Then the formula (*) is true.

5 Higher Chow groups

Let X be a smooth complex projective variety of dimension n . Consider the group $C^k(X, 1)$ which is the free abelian group generated by pairs (W, f) , where W is an irreducible algebraic subvariety of X of codimension $k-1$ and f is a non-zero rational function on W , modulo the relations

$$(W, f_1 f_2) = (W, f_1) + (W, f_2),$$

where f_1 and f_2 are two rational functions on W . The group $Z^k(X, 1)$ is defined to be the kernel of the map $C^k(X, 1) \rightarrow Z^k(X)$ sending (W, f) to $\text{div } f$, the divisor of f . Define the group $B^k(X, 1)$ as the subgroup of $Z^k(X, 1)$ generated by elements of the form

$$(\text{div } g, h|_{\text{div } g}) - (\text{div } h, g|_{\text{div } h}),$$

where g, h are non-zero rational functions on some $V \subset X$ of codimension $k-2$ whose divisors have no component in common.

The first higher Chow group is defined as follows:

$$CH^k(X, 1) := Z^k(X, 1) / B^k(X, 1).$$

part 3

6 Abel-Jacobi map

We use the definition of the Abel-Jacobi map from [3]. Let x represent an element of $CH^k(X, 1)$, $k \in Z^i(X, 1)$ i.e.

$$x = \sum_i (W_i, f_i), \quad \sum_i \text{div } f_i = 0.$$

We denote the corresponding element in $CH^k(X, 1)$ by $[x]$. We choose a path $[0, \infty] \subset \mathbb{CP}^1$. Let us denote $\gamma_i = f_i^*[0, \infty]$, which is a $2n - 2k + 1$ -chain on X whose boundary is $-\text{div } f_i$. This implies that the chain $\gamma = \sum_i \gamma_i$ is a cycle. By Poincaré duality γ has a class $[\gamma] \in H^{2k-1}(X, \mathbb{Z})$. Using Hodge theory one can prove that this class is torsion. Suppose for simplicity that the class is trivial, $\gamma = \partial\eta$ for some $2n - 2k + 2$ -chain η . Then for any $\omega \in F^{n-k+1, 2n-2k+2}(X)$ with $d\omega = 0$ we consider the following expression:

$$\langle AJ^{k,1}[x], [\omega] \rangle = \frac{1}{2\pi i} \sum_i \int_{W_i \setminus \gamma_i} \omega \log f_i + \int_\xi \omega,$$

where the logarithm on \mathbb{CP}^1 is defined using the cut along the chosen path $[0, \infty]$. This defines a map

$$AJ^{k,1} : CH^k(X, 1) \rightarrow \frac{H^{2k-2}(X, \mathbb{C})}{F^k H^{2k-2}(X, \mathbb{C}) + H^{2k-2}(X, \mathbb{Z})}.$$

7 Families of cycles and modular forms

We come back to our family \mathcal{X} . Suppose we have subvarieties $W_i \subset \mathcal{X}$ with rational functions f_i on them such that the total divisor $\sum_i \text{div } f_i$ is vertical. This gives a family $x_t \in CH^k(X_t, 1)$ ($t \in S_0$, S_0 is Zariski open in S). Taking the Abel-Jacobi map we obtain a family

$$\epsilon_t \in \frac{\text{Hom}_{\mathbb{C}}(F^{k-1} H^{2k-2}(X_t, \mathbb{C}), \mathbb{C})}{H^{2k-2}(X_t, \mathbb{Z})}.$$

Let \mathcal{D}^j denote the sheaf of rings of differential operators on S of degree at most j . Let M^j denote the sheaf of modular forms on S of weight j . There is a canonical section B of the sheaf $M^{2k} \otimes \mathcal{D}^{2k-1} \otimes M^{2k-2}$. Moreover, this section goes to zero if we apply the Gauss-Manin connection map $\mathcal{D} \otimes H^{2k-2} \rightarrow H^{2k-2}$ (here H^{2k-2} denotes the cohomology sheaf on S of the family \mathcal{X} and $M^{2k-2} = F^{2k-2} H^{2k-2}$). This section is constructed as follows. On a small Zariski open set U of S we can take a modular form f of weight $2k-2$ and a modular form g of weight $2k$ such that both are not zero on U . The following expression is a differential operator on U

$$D(g, f) = \frac{1}{g} \left(\frac{\partial}{\partial z} \right)^{2k-1} \frac{1}{f}$$

because it maps modular functions to modular functions. We put $B(U) = g \otimes D(g, f) \otimes f$. We can multiply B with ϵ_t and get a meromorphic modular form of weight $2k$, $B\epsilon \in M^{2k}(S_0)$. Therefore we obtain a map from families of elements of $CH^k(X_t, 1)$ to modular forms of weight $2k$. In fact in my thesis I compute this map in a purely algebraic way in terms of iterated residues.

part 4

8 All objects together

There is also a way to obtain a modular form of weight $2k$ from the Green's function. We fix one of the variables, say z_1 and consider $G_k^\Gamma(z_1, z)$ as a function of z . Then the k -th non-holomorphic derivative g of $G_k^\Gamma(z_1, z)$ is a meromorphic modular form of weight $2k$. In fact it is uniquely determined by the properties:

- It is a meromorphic modular form of weight $2k$ in z , whose set of poles is Γz_1 , which is zero at the cusps. The pole at z_1 is of the type $\left(\frac{z_1 - \bar{z}_1}{(z - z_1)(z - \bar{z}_1)} \right)^k$.
- Its cohomology class is totally real for k even and totally imaginary for k odd.

Theorem 1 Suppose one can construct a family of elements of $CH^k(X_t, 1)$ such that the corresponding modular form is g . Then the conjecture is true for this fixed z_1 and arbitrary CM point z_2 .

For the point $z_1 = \sqrt{-1}$, $\Gamma = PSL_2(\mathbb{Z})$, $k = 2$ we have

$$g = E_4 \frac{864j - 1492292}{(j - 1728)^2}.$$

In my thesis I explicitly construct a family of elements of $CH^2(E_t \times E_t, 1)$ for the family $y^2 = x^3 - x - t$. I compute the corresponding modular form and obtain exactly g , thereby proving a theorem

Theorem 2 The conjecture is true for the case $\Gamma = PSL_2\mathbb{Z}$, $k = 2$, $z_1 = \sqrt{-1}$ and z_2 — arbitrary CM point.

In particular I prove the first identity from the Examples section.

References

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