Multiple Zeta Values and the Connes-Moscovici Hopf Algebra

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part 1

Multiple Zeta Values

For an n-tuple of positive integers \( k = (k_1, k_2, \ldots, k_n) \) with \( k_i > 1 \), we define multiple zeta values \( \zeta(k) \) by the convergent series
\[
\zeta(k) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha_1^k_1 \alpha_2^k_2 \cdots \alpha_n^k_n}
\]
MZV has an integral representation as follows.
\[
\zeta(k) = \int_0^1 \cdots \int_0^1 \omega(t_1, \omega(t_2, \cdots \omega(t_n)) dt_1 \cdots dt_n
\]
where \( \omega(t) = t_1 + t_2 + \cdots + t_n \) is the weight of \( \zeta(k) \) and \( \omega(t) = 0 \) if \( n > 1 \).

There is a little known about arithmetical properties of these numbers. On the other hand there is a huge amount of research (linear and algebraic) among MZV’s. One of the fascinating features of MZV’s is that the structure of relations over \( Q \) reflects other structures in mathematics and physics.

part 2

The Modular Algebra

Let \( M \) be the ring of modular forms of all levels and all weights. \( M \) is graded by the weight and the group \( \text{PGL}_2(\mathbb{Q}) \) acts on \( M \) via the scalar operator
\[
f_c(x) = (dx + b)^m (cx + d)^n f(x)
\]
where \( k \) is the weight of \( f \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}_2(\mathbb{Q}) \). The modular product \( \text{A} \times \text{PGL}_2(\mathbb{Q}) = \text{Modular algebra} \).

part 3

CM-action

Theorem. For an \( n \)-tuple of positive integers \( k = (k_1, k_2, \ldots, k_n) \) with \( k_i > 1 \), we define multiple zeta values \( \zeta(k) \) by the convergent series
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part 4

Connection to MZV’s

Let \( L \) be the ring of the quasi-free algebra on the letters \( X, \theta \) by the relations that all \( M_x X(\theta) \theta M_x \) commute. Alternatively, \( L \) is the Lie algebra with basis \( \{X, \theta, \delta_1, \ldots, \delta_n\} \) and relations \( [X, \theta] = \theta [X, \theta] = 0 \).

Let \( \zeta(k) \) be the weight of \( \zeta(k) \) and the group \( A \) acts on \( M \) by
\[
f_c(x) = (dx + b)^m (cx + d)^n f(x)
\]
where \( k \) is the weight of \( f \) and \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PGL}_2(\mathbb{Q}) \). The modular product \( \text{A} \times \text{PGL}_2(\mathbb{Q}) = \text{Modular algebra} \).

References

3. Gómez, Héctor; Katz, Mikhail; Zagier, Don Double zeta values and modular forms, Analytic number theory and automorphic forms, 75–106.