Hypersurfaces and motives from Feynman graphs

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Introduction

Given a Feynman graph Γ one associates a variety X_Γ in the following way. To each edge e of Γ, assign a formal variable x_e and define the *Kirchhoff polynomial* of Γ

$$\Psi_\Gamma := \sum_T \prod_{e \notin T} x_e,$$

where T is a spanning tree of Γ. It follows from the definition that Ψ_Γ is homogeneous of degree equal to the rank of $H_1(\Gamma, \mathbb{Z})$ (called the *loop number* in physics terminology) and linear in each variable.

Definition 1. The *graph variety* $X_\Gamma := \{\Psi_\Gamma = 0\}$. If Γ has n edges then $X_\Gamma \subseteq \mathbb{P}^{n-1}$ in which case we call X_Γ the *graph hypersurface*.

Note that in general X_Γ is a singular variety and understanding the singular loci of X_Γ explicitly remains an important task. However, for the wheel-with-n-spokes graph, in §4 of [7], the authors show that the graph hypersurface admits a natural resolution of singularities. The relationship between graph hypersurfaces and motives began with an unpublished conjecture of Kontsevich (apparently based on [8, 9] and the Weil conjectures) that for $q = p^n$ where p is prime, if the number of $\mathbb{F}_q$-points of X_Γ is a polynomial in q then $[X_\Gamma]$ generates the full Grothendieck group of mixed Tate motives when we run over all Γ. (Recall that a mixed Tate motive is either a direct sum of an arbitrary twist of a pure Tate motive $\mathbb{Q}(n)$ or an successive 1-extension in the triangulated derived category of mixed motives.)
This conjecture was disproved by Belkale-Brosnan who showed that $[X_\Gamma]$ additively generates the Denef-Loeser ring of geometric motives [4]. In other words, for arbitrary Γ, one expects to find a motive $[X_\Gamma]$ that is more general than the mixed Tate type. Nevertheless, in a seminal paper [7] Bloch-Esnault-Kreimer prove the following

Theorem 2. *Let Γ be the wheel-with-n-spokes graph with the graph hypersurface X_n. Then*

$$H^{2n-1}(\mathbb{P}^{2n-1} \backslash X_n) \simeq \mathbb{Q}(2n-3).$$

Equivalently,

$$H^{2n-1}(X_n, \mathbb{Q})_{prim} \simeq \mathbb{Q}(-2)$$

where $H^{2n-1}(X_n, \mathbb{Q})_{prim} := \mathrm{coker}(L : H^{2n-1}(X_n, \mathbb{Q}) \longrightarrow H^{2n}(X_n, \mathbb{Q}))$ and L is the Lefschetz operator on cohomology.

The arithmetic significance of this theorem, very roughly speaking, is as follows (for important details, see [6, 7].) Let σ be a simplex in $\mathbb{P}^{2n-1}_{\mathbb{R}}$ with the boundary of σ supported on the standard coordinate simplex. We know from physics computations [8] that the period evaluates as (for $n \geq 3$)

$$\int_\sigma \frac{\Omega}{\Psi_\Gamma^2} = \frac{c}{\pi^{2n}} \zeta(2n-3),$$

where $c \in \mathbb{Q}^\times$ and $\Omega := \sum_{i=1}^{2n} (-1)^i x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{2n}$ is a differential form of top degree. Ignoring several serious subleties for which we refer the reader to [6], we can conclude that the form $\frac{\Omega}{\Psi_\Gamma^2}$ lies in $\mathbb{Q}(2n-3)_{\mathrm{DR}}$ (the subscript denotes that we are working with the deRham realization of the motive.) Now from a general conjecture due to Goncharov (see chapter 25 of [2]) we know that periods of mixed Tate motives should be (multiple) zeta values. The theorem of Bloch-Esnault-Kreimer is a verification of a very special case of this conjecture.

Insertion of graphs and singularities of graph varieties

The following work is a joint project with C. Bergbauer and is being written up in [5]. Notation: Let G be a (Feynman) graph. E_G and V_G denotes the set of edges and vertices of G respectively. By E_G^{ext} we mean the set of all open edges of G (i.e. edges that have only one vertex). By E_G^{int} we mean the set of all edges that begin and end in vertices. We sometimes refer to such edges as "internal".
In this section, I describe some partial results that have been obtained about understanding how singularities of graph hypersurfaces change when a graph is inserted into another graph. Let γ and Γ be two connected graphs. Fix some vertex $v \in V_\Gamma$ and define

$$E_v := \{\text{set of all edges of } \Gamma \text{ that are adjacent to } v\}.$$

Definition 3. An *insertion of γ into* Γ is denoted as $\Gamma \leftarrow \gamma$ and is obtained by removing v and identifying the open edges of Γ thus obtained to that of γ via the one-to-one glueing map $s : E_v \longrightarrow E_\gamma^{\mathrm{ext}}$.

Using this we define

Definition 4. The *pointed Kirchhoff polynomial* $\widehat{\Psi_{\Gamma,\gamma}}$ is defined in the following way:

(1) (equivalence of edges) If $e, e' \in E_v$ is part of a cycle of Γ, then write $e \sim e'$.
(2) (definition of $\gamma_{e_i,e_i'}$) Let $e_i \sim e_i'$. The graph $\gamma_{e_i,e_i'}$ is the graph γ where, for all i, the open edges $s(e_i)$ and $s(e_i')$ are joined to form a new internal edge. We denote this new internal edge as $e_i \circ e_i'$.
(3) (definition of $\Gamma_{e_i,e_i'}$) Split $v \in E_v$ in the sense that e_i and e_i' end in different copies of v. The graph $\Gamma_{e_i,e_i'}$ is obtained from Γ in this way.

Then

$$\widehat{\Psi_{\Gamma,\gamma}} := \sum_{e_i \sim e_i'} \Psi_{\gamma_{e_i,e_i'}} \Psi_{\Gamma_{e_i,e_i'}} \mod (e_i \circ e_i').$$

With definition 4 at hand, we have

Proposition 5. $\Psi_{\Gamma \leftarrow \gamma} = \Psi_\Gamma \Psi_\gamma + \widehat{\Psi_{\Gamma,\gamma}}$.

The reader is invited to compare proposition 5 to proposition 3.5 of [7].
As an immediate and elementary application of proposition 5, we have

Proposition 6.

$$X_\gamma \cap X_\Gamma \cap \mathit{Sing}\ \widehat{X_{\Gamma,\gamma}} \subseteq \mathit{Sing}\ X_{\Gamma \leftarrow \gamma},$$

where $\widehat{X_{\Gamma,\gamma}} := \{\widehat{\Psi_{\Gamma,\gamma}} = 0\}$ and Sing denotes the singular locus.

The next step in this project is to see under what restrictions on Γ and γ is the above inclusion actually an equality. We would also like to clarify what happens when we symmetrize the construction in definition 4, i.e., instead of fixing some v we should try to carry the construction for all possible $v \in E_\Gamma$ and sum over it. We would also like to exhibit that there exists a natural resolution of singularities for the variety $X_{\Gamma \leftarrow \gamma}$. The idea is to use the methods of §4 of [7].

Lie and Hopf algebras of graph motives

It is well known by now that the physical procedure of renormalization, i.e. extracting finite values from divergent Feynman integrals, is combinatorially encoded in a Hopf algebra of Feynman graphs (due to Kreimer and Connes-Kreimer, cf, [10].) Furthermore it is also well known that in certain cases finite values of these divergent integrals (the so-called "residue" of the corresponding Feynman graph) evaluate to multiple zeta values (MZVs) [9]. Now the space of MZVs themselves has a Hopf algebra structure through shuffle and concatenation products on symbols related to Chen integrals and this Hopf algebra can be given a motivic interpretation [11]. It is also known, thanks to [11], that (framed) mixed Tate motives form a Hopf algebra. We have been investigating ways in which all of these Hopf algebras maybe compatible. I have been working on an expository paper [14] where I study all of the Hopf algebras mentioned above side-by-side. The driving mental picture is

$$\begin{array}{ccccc}
X_\Gamma & \longrightarrow & H^*(X_\Gamma) & \longrightarrow & \mathcal{MTM} \\
\uparrow & & \downarrow & & \| \\
\Gamma & \longrightarrow & \mathrm{MZV} & \longleftarrow & \mathcal{MTM} \\
\downarrow & & \downarrow & & \downarrow \\
\Gamma \otimes \Gamma & \longrightarrow & \mathrm{MZV} \otimes \mathrm{MZV} & \longleftarrow & \mathcal{MTM} \otimes \mathcal{MTM}
\end{array}$$

Conceptually there is no strong obstacle in seeing the compatibility between the different coproducts. For example, all coproducts arising in the aforementioned picture are commutative but not cocommutative. For a somewhat different perspective on this problem, see [6].
Another approach to this problem is the following.

(1) Construct a Lie algebra of insertions as defined in definition 3 based on a pre-Lie operation $\Gamma \leftarrow \gamma$. This should be related to the Connes-Kreimer Lie algebra [10].
(2) Lift this Lie algebra to a Lie algebra of pointed Kirchoff polynomials.
(3) Use the Cartier-Milnor-Moore theorem and the methods of §7 of [6] to obtain a Hopf algebra of graph motives. (This is a coproduct morphism in the category of graph motives.) In particular, check whether we obtain the morphism 7.7 of [6]:

$$M(\Gamma) = H^{m-1}(\mathbb{P}^{m-1} \backslash X_\Gamma, \mathbb{Q}(m-1)) \longrightarrow H^{m-2}(E \backslash E \cap Y, \mathbb{Q}(m-2)) \simeq M(\Gamma') \otimes M(\Gamma//\Gamma').$$

Here Y is the strict transform of X_Γ and E the exceptional divisor obtained by blowing up the faces of the coordinate simplex.

In relation to this, we would also like to investigate general properties of mixed Tate categories [11] that have Lie algebra morphisms.
A key problem in understanding motivic aspects of Feynman graphs is the lack of examples. (So far one only has one rigorous theorem in this direction, namely for the wheel-with-n spokes case, theorem 2.) I have been thinking of ways that one can generate more examples. The basic technology is from [12]. The idea is this: in general, constructing mixed Hodge structures for singular projective varieties is a convoluted problem. (See [3] for this.) However, when the singularities of the variety at hand are *normal crossing*, the methods of [12] can be made fairly explicit, as demonstrated in [13]. Furthermore, the construction of cohomological mixed Hodge structure for such varieties is functorial. Using these methods, it seems that one should (a) first automate checking whether a given graph variety has normal crossing singularities or not and (b) use the methods of [12] to construct the mixed Hodge stuctures. Now it is known that for number fields, the Hodge realization functor (from mixed Tate motives to mixed Hodge-Tate structures) is exact and faithful. Therefore, having the mixed Hodge structures for graph varieties will enable us to "guess" the corresponding mixed Tate motives. For step (a), the idea is to augment the program we already have for generating singular loci of graph varieties to also include checking for normal crossing singularities. Step (b) of course remains quite challenging.

A related question

It was also recently pointed out to me by P. Aluffi that computing the topological Euler characteristics of singular graph hypersurfaces may also be worthwhile since (very roughly speaking) it measures how "bad" the singularities of the variety are. Aluffi has written a computer program in MACAULAY that implements this [1]. It would be interesting to see if we can integrate a code Bergbauer and I have written (which explicitly lists the singularities of X_Γ for a given Γ) with his routine and compute the topological Euler characteristics of graph varieties with arithmetically interesting periods. Consider Γ_1 and Γ_2 evaluating to $\zeta(k)$ and $\zeta(k+l)$ respectively for a fixed $k \in \mathbb{N}$ and varying $l > k$.

Question 7. Does the topological Euler characteristic of X_{Γ_1} differ from X_{Γ_2}? If so, does the difference depend on l?

The main obstacle to large-scale computer experimentation seems to be the slow run-time of both routines.

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